Equalization of longitudinal and transverse beam cooling rates due to strong synchro-betatron interactions

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This paper shows that longitudinal and transverse beam cooling rates can be equalized by strong synchro-betatron interactions on a synchro-betratron sum resonance or in its neighborhood in a general beam cooling system in circular particle accelerators. This equalizing phenomenon enhances brightness for large synchrotron radiation sources, permits simultaneous stochastic cooling of a bunched beam in longitudinal and transverse directions, and allows longitudinal and transverse laser cooling of a relativistic ion bunched beam.

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I. INTRODUCTION

Longitudinal beam cooling means damping of synchrotron oscillations in a longitudinal plane, while transverse beam cooling means damping of betatron oscillations in a transverse plane. In 1958, Robinson [1] showed that the damping rates of synchrotron and betatron oscillations cannot be changed by any form of external electromagnetic fields. Recently Csonka [2] presented a more general proof of this theorem. Robinson's theorem is acceptable only under the condition of negligible synchro-betatron interactions.

We have clarified how the synchro-betatron interactions affect the damping rates of the betatron and synchrotron oscillations where there is no coupling between horizontal and vertical betatron oscillations. We have investigated strong synchro-betatron interactions which occur on a synchro-betatron resonance or in its neighborhood due to either a longitudinal kick by an rf electric field in an accelerating cavity with a finite horizontal dispersion, or a transverse kick by an rf magnetic field in a deflecting cavity. Our theory shows that the damping rates of synchrotron and betatron oscillations can be changed due to betatron modulation of synchrotron oscillations generated by synchro-betatron interactions so as to have a tendency to equal each other. This means that longitudinal and transverse beam cooling rates can be equalized by synchro-betatron interactions. This equalization of beam cooling rates can be applied to any kind of beam cooling system. As already known, similar features can be found in intrabeam scattering [3], where kinetic-energy transfer between transverse and longitudinal motions depends on multiparticle dynamics. Contrarily kinetic-energy transfer by synchro-betatron interactions is determined by single-particle dynamics. Thus the equalization of beam cooling rates by synchro-betatron interactions is independent of particle density and so is very easily controllable.

In Sec. II, we derive the betatron modulation of the synchrotron oscillations. In Sec. III, we present effects on synchrotron and betatron emittances in a general cooling system. In Sec. IV, we show a particle simulation to support our theory and discuss some applications of the equalization of longitudinal and transverse beam cooling rates.

II. BETATRON MODULATION OF SYNCHROTRON OSCILLATIONS

Synchro-betatron interactions are driven by dispersion in rf cavities, deflecting fields, and so on. In the following, we summarize their mechanism of generation.

For synchro-betatron interactions driven by dispersion in rf cavities, the instantaneous position and angle of the particle do not change by rf acceleration, but the equilibrium orbit of the particle suddenly changes due to the energy change through the rf acceleration, which leads to changes of the amplitude and phase of the betatron oscillations. The rf acceleration depends on the longitudinal position. This means that synchrotron oscillations affect the betatron oscillations. Furthermore, betatron oscillations change the orbit length per revolution, which leads to a change of the longitudinal position. Thus the betatron oscillations also affect the synchrotron oscillations.

For synchro-betatron interactions driven by deflecting fields, the transverse deflection depends on the longitudinal position of the particle, and thus the synchrotron oscillations affect the betatron oscillations. Furthermore, the rf acceleration depends on the transverse position of the particle, and thus the betatron oscillations affect the synchrotron oscillations.

The interactions driven by dispersion in rf cavities were first analyzed by Piwinski and Wrulich [4]. They considered the change of the orbit length due to the betatron oscillations, and clarified the sum and difference resonance features. The interactions driven by deflecting fields were first studied by Sundelin [5] in a preliminary analytical formulation, and later extensively studied by Suzuki [6] using a canonical perturbation theory. In this paper, we present our own original method for analyzing the synchro-betatron interactions. Here we assume zero chromaticity and negligible synchrotron modulation of betatron oscillations for a convenient theoretical treatment. Then we first derive betatron modulation of synchrotron oscillations in this section.

Properly we should develop our theory in a dissipative system, but a state in which betatron and synchrotron oscillations are very slowly damped over a large number of revolutions permits analysis using a Hamiltonian in a conservative system. Then we consider the conservative system initially to describe the betatron modulation of synchrotron oscillations. The synchrotron oscillations can be described with a pair of canonical variables (σ,δ), where σ is a longitudinal advance in meters from an equilibrium particle and δ is a relative energy deviation [7]. Instead of σ, for convenience we use the variable ϕ defined as $\phi=-k\sigma/\beta_0$, with β_0 being the relativistic velocity of the particle and k the wave number of the accelerating field ($k=2\pi f/c$, for an accelerating frequency f and velocity of light c.)

A Hamiltonian H can be developed into a Fourier series of a betatron phase θ_0 at a certain longitudinal point ($s=0$) to be described as

$$H=b_0+\sum_{n=1}^{\infty}(\bar{a}_n \sin n\theta_0+\bar{b}_n \cos n\theta_0) , \tag{1}$$

where $\bar{a}_n=a_n\cos n\mu_x-b_n\sin n\mu_x$, $\bar{b}_n=a_n\sin n\mu_x+b_n\cos n\mu_x$, and μ_x is the betatron phase advance from 0 to s. Betatron-modulated components of the synchrotron oscillations δ_β and ϕ_β approximately satisfy the differential equations

$$\frac{d\delta_\beta}{ds}=k\sum_{n-1}^{\infty}(\bar{a}_n^\phi \sin n\theta_0+\bar{b}_n^\phi \cos n\theta_0) , \tag{2}$$

$$\frac{d\phi_\beta}{ds}=-kb_0^{\delta\delta}\delta_\beta-k\sum_{n=1}^{\infty}(\bar{a}_n^\delta \sin n\theta_0+\bar{b}_n^\delta \cos n\theta_0) , \tag{3}$$

where the superscripts δ and ϕ indicate partial derivatives with respect to δ and ϕ, respectively, and $b_0^{\delta\delta}$ is a second partial derivative of b_0 with respect to δ. Supposing that the betatron oscillations are simple damping oscillations, Eqs. (2) and (3) are integrated to give

$$\delta_\beta=k\sum_{n=1}^{\infty}(\underline{A}_n^\phi \sin n\theta_0+\underline{B}_n^\phi \cos n\theta_0) , \tag{4}$$

$$\phi_\beta=-k\sum_{n=1}^{\infty}[(\underline{A}_n^\delta+k\underline{\alpha}_n)\sin n\theta_0+(\underline{B}_n^\delta+k\underline{\beta}_n)\cos n\theta_0)] , \tag{5}$$

where

$$\underline{A}_n^\phi=\bar{\bar{A}}_n^\phi+t_n\hat{B}_n^\phi-\tfrac{1}{2}\hat{A}_n^\phi , \tag{6}$$

$$\underline{B}_n^\phi=\bar{\bar{B}}_n^\phi-t_n\hat{A}_n^\phi-\tfrac{1}{2}\hat{B}_n^\phi , \tag{7}$$

$$\hat{A}_n^\phi=A_n^\phi C_{1n}-B_n^\phi S_{1n} , \tag{8}$$

$$\hat{B}_n^\phi=A_n^\phi S_{1n}+B_n^\phi C_{1n} , \tag{9}$$

$$\bar{\bar{A}}_n^\phi=\int_0^s \bar{a}_n^\phi ds' , \quad \bar{\bar{B}}_n^\phi=\int_0^s \bar{b}_n^\phi ds' , \tag{10}$$

$$A_n^\phi=\oint \bar{a}_n^\phi ds , \quad B_n^\phi=\oint \bar{b}_n^\phi ds , \tag{11}$$

$$\underline{\alpha}=\bar{\bar{\alpha}}_n+t_n\hat{\beta}_n-\tfrac{1}{2}\hat{\alpha}_n , \tag{12}$$

$$\underline{\beta}_n=\bar{\bar{\beta}}_n-t_n\hat{\alpha}_n-\tfrac{1}{2}\hat{\beta}_n , \tag{13}$$

$$\bar{\bar{\alpha}}_n=\int_0^s b_0^{\delta\delta}(\bar{\bar{A}}_n^\phi+\dot{A}_n^\phi)ds' , \quad \bar{\bar{\beta}}_n=\int_0^s b_0^{\delta\delta}(\bar{\bar{B}}_n^\phi+\dot{B}_n^\phi)ds' , \tag{14}$$

$$\hat{\alpha}_n=\langle b_0^{\delta\delta},\bar{a}_n^\phi\rangle C_{1n}-\langle b_0^{\delta\delta},\bar{b}_n^\phi\rangle S_{1n} +B_0^{\delta\delta}(\dot{A}_n^\phi C_{2n}-\dot{B}_n^\phi S_{2n}) , \tag{15}$$

$$\hat{\beta}_n=\langle b_0^{\delta\delta},\bar{a}_n^\phi\rangle S_{1n}+\langle b_0^{\delta\delta},\bar{b}_n^\phi\rangle C_{1n} +B_0^{\delta\delta}(\dot{A}_n^\phi S_{2n}+\dot{B}_n^\phi C_{2n}) , \tag{16}$$

$$\dot{A}_n^\phi=t_nB_n^\phi-\tfrac{1}{2}A_n^\phi , \quad \dot{B}_n^\phi=-t_nA_n^\phi-\tfrac{1}{2}B_n^\phi , \tag{17}$$

$$t_n=\tfrac{1}{2}\cot(\pi nq_x) , \tag{18}$$

$$B_0^{\delta\delta}=\oint b_0^{\delta\delta}ds , \tag{19}$$

$$S_{1n}=-\gamma t_n , \quad C_{1n}=1+\frac{\gamma}{2} , \tag{20}$$

$$S_{2n}=2S_{1n}C_{1n} , \quad C_{2n}=C_{1n}^2-S_{1n}^2 , \tag{21}$$

$$\gamma=\gamma_x-\gamma_\sigma , \tag{22}$$

where q_x is the fractional part of a betatron tune $(-0.5\le q_x<0.5)$, γ_x is the damping rate of a betatron amplitude, and γ_σ is the damping rate of a synchrotron amplitude. The quantities $\underline{A}_n^\delta$ and $\underline{B}_n^\delta$ are similarly defined where the superscripts ϕ are replaced by δ in Eqs. (6)–(11), $\oint ds$ indicates an integration along a closed orbit, and the brackets $\langle\ ,\ \rangle$ indicate a double integral

$$\langle f,g\rangle=\oint f\int_0^s g ds' ds , \tag{23}$$

with f and g as arbitrary continuous functions of s. The betatron modulations δ_β and ϕ_β can be used approximately in a dissipative system as described before.

III. EFFECTS ON SYNCHROTRON AND BETATRON EMITTANCES

The betatron motions are expressed with a pair of canonical variables (x,p_x), where x is a transverse displacement and p_x is a canonical momentum of x which is defined as

$$p_x=\beta_0^2\frac{P_x}{P_0} , \tag{24}$$

with β_0 as a relativistic velocity, P_x as the transverse kinetic momentum, and P_0 as the total kinetic momentum [7]. The change of the betatron emittance per revolution, $\Delta\varepsilon_x$, can be written as

$$\Delta\varepsilon_x=\oint\left[-2\frac{\partial H}{\partial\theta}+\frac{2}{\beta_x^2}(\alpha_x'\beta_x-\alpha_x\beta_x')xy +\frac{\beta_x'}{\beta_x}\left[\frac{2}{\beta_x}y^2-\varepsilon_x\right]\right]ds , \tag{25}$$

where the prime indicates a derivative with respect to a longitudinal coordinate s, α_x and β_x are optical functions referred to as Twiss parameters, and $y=\alpha_x x+\beta_x p_x$. Equation (25) is averaged over several betatron oscillations to give the average change of the betatron emittance per revolution

$$\langle \Delta\varepsilon_x \rangle_\theta = -2\left\langle \oint \frac{\partial H}{\partial \theta} ds \right\rangle_\theta . \tag{26}$$

The kernel of the integral, $\partial H/\partial\theta$, is developed with respect to the betatron modulation of synchrotron oscillations to give

$$\frac{\partial H}{\partial \theta} = \sum_{n=1}^{\infty} n(G_n^0 + G_n^\delta \delta_\beta + G_n^\phi \phi_\beta) , \tag{27}$$

where

$$G_n^0 = \bar{a}_n \cos n\theta_0 - \bar{b}_n \sin n\theta_0 , \tag{28}$$

G_n^δ and G_n^ϕ indicate partial derivatives of G_n^0 with respect to δ and ϕ, respectively. The term G_n^0 is integrated over one revolution to become zero. Taking only the first term ($n=1$) as a main term, Eq. (26) becomes

$$\langle \Delta\varepsilon_x \rangle_\theta = -2\oint \Gamma_0 ds , \tag{29}$$

where

$$\Gamma_0 = \langle G_1^\delta \delta_\beta + G_1^\phi \phi_\beta \rangle_\theta , \tag{30}$$

$$G_1^\delta = \bar{a}_1^\delta \cos\theta_0 - \bar{b}_1^\delta \sin\theta_0 , \tag{31}$$

$$G_1^\phi = \bar{a}_1^\phi \cos\theta_0 - \bar{b}_1^\phi \sin\theta_0 . \tag{32}$$

Using the main term with $n=1$ of Eqs. (4) and (5), which are the expressions of betatron modulations of synchrotron oscillations, we obtain

$$\begin{aligned}\langle \Delta\varepsilon_x \rangle_\theta = &-2\varepsilon_x a(1+2tS_1 - C_1) \\ &+2\varepsilon_x b[tC_1 + \tfrac{1}{2}S_1 + (T-\tfrac{1}{2})S_2 - tC_2] \\ &-2\varepsilon_x \gamma_{xd} ,\end{aligned} \tag{33}$$

where

$$\begin{aligned}2\varepsilon_x a = k(&A_1^\delta B_1^\phi - A_1^\phi B_1^\delta) \\ &+k^2(A_1^\phi \langle \bar{b}_1^\phi, b_0^{\delta\delta} \rangle - B_1^\phi \langle \bar{a}_1^\phi, b_0^{\delta\delta} \rangle) ,\end{aligned} \tag{34}$$

$$2\varepsilon_x b = -k^2 B_0^{\delta\delta}[(A_1^\phi)^2 + (B_1^\phi)^2] , \tag{35}$$

$$t = \tfrac{1}{2}\cot(\pi q_x) , \tag{36}$$

$$T = t^2 + \tfrac{1}{4} , \tag{37}$$

S_{1n}, C_{1n}, S_{2n}, and C_{2n} for $n=1$ are replaced by S_1, C_1, S_2, and C_2, respectively. Here a dissipative damping effect is introduced as the last term in Eq. (33), where γ_{xd} is the damping rate of the betatron amplitude. When the synchrotron and betatron emittances are constant with no dissipative damping force, i.e., $\gamma_x = \gamma_\sigma = 0$ ($\gamma = 0$), so that particles execute a simple periodic betatron oscillation, we have $S_1 = 0$ and $C_1 = 1$ from Eq. (20) with the self-consistent result that the right-hand side of Eq. (33) vanishes. In a dissipative system with $\gamma_{xd} \neq \gamma_{\sigma d}$ ($\gamma_{\sigma d}$ is the damping rate of the synchrotron amplitude), however, we have the result that $S_1 \neq 0$ and $C_1 \neq 1$ on an average over several revolutions. Then we obtain the important result that electromagnetic fields can change the damping rates of synchrotron and betatron oscillations due to synchro-betatron interactions in the dissipative system.

We show the damping region of betatron oscillations in the space (C_1, S_1) in the dissipative system. The space (C_1, S_1) is rotated by πq_x around the origin to generate a new space (C, S), where Eq. (33) can be rewritten as

$$\left\langle \frac{\Delta\varepsilon_x}{\varepsilon_x} \right\rangle_\theta = \frac{b}{S_0^2}(C - \lambda S_0)(S + S_0) - 2\gamma_{xd} . \tag{38}$$

where

$$S_0 = \sin(\pi q_x) , \quad \lambda = \frac{2a}{b} . \tag{39}$$

Then we obtain the damping region of the betatron oscillations:

$$(C - \lambda S_0)(S + S_0) < 2\left|\frac{\gamma_{xd} S_0^2}{b}\right| . \tag{40}$$

Next we consider the damping rate of synchrotron oscillations. The synchrotron emittance ε_σ can be written as

$$\varepsilon_\sigma = \beta_\sigma \delta^2 - \frac{2}{k^2 \beta_\sigma \cos\phi_{sa}}[\cos(\phi_{sa} + \phi) + \phi \sin\phi_{sa} - \cos\phi_{sa}] , \tag{41}$$

where ϕ_{sa} is the synchronous phase of the accelerating field, and β_σ is the betatron function of the synchrotron oscillations [8] written as

$$\beta_\sigma \cong \frac{\eta_m L}{2\pi \beta_0 \nu_s} \tag{42}$$

where L is the circumference of the ring, ν_s the synchrotron tune,

$$\eta_m = \alpha_m - \frac{1}{\gamma_0^2} , \tag{43}$$

and α_m is the momentum compaction factor. Taking into account a damping rate $\gamma_{\sigma d}$ due to the dissipative force, the rate of change of the synchrotron emittance per revolution averaged over several betatron oscillations becomes

$$\langle \Delta\varepsilon_\sigma \rangle_\theta = \left|\frac{b\varepsilon_x}{2\pi\beta_0 \nu_s}\right|(1 + 2tS_1 - C_1) - 2\gamma_{\sigma d}\varepsilon_\sigma . \tag{44}$$

Therefore we obtain the damping region of the synchrotron oscillations:

$$2tS_1 - C_1 < 4\pi\gamma_{\sigma d}\left|\frac{\varepsilon_\sigma}{\varepsilon_x}\right|\left|\frac{\beta_0 \nu_s}{b}\right| - 1 . \tag{45}$$

As shown in Eqs. (38) and (44), the rates of change of the betatron and synchrotron emittances per revolution depend on the quantities a and b, which are determined by $B_0^{\delta\delta}$, A_1^ϕ, A_1^δ, B_1^ϕ, B_1^δ, $\langle \bar{a}_1^\phi, b_0^{\delta\delta} \rangle$, and $\langle \bar{b}_1^\phi, b_0^{\delta\delta} \rangle$ obtained from $b_0^{\delta\delta}$, a_1^ϕ, b_1^ϕ, a_1^δ, and b_1^δ. (See the Appendix about the expressions of these quantities.)

Figure 1 shows the damping regions of the betatron and synchrotron oscillations expressed in Eqs. (40) and (45) as hatched areas, while the other spaces are growing regions. This figure indicates the case where the fractional part of the betatron tune q_x is negative, because the synchro-betatron sum resonance is important, as described below. The boundary line of the damping region is a hyperbola for the betatron oscillations, while it is a straight line for the synchrotron oscillations. The effects of the dissipative damping rates γ_{xd} and $\gamma_{\sigma d}$ are exaggerated in Fig. 1 in order to clarify those effects; in practice the boundaries of the damping regions are in the neighborhood of the broken lines.

In the conservative system with $\gamma_{xd}=\gamma_{\sigma d}=0$, the boundaries of the damping regions coincide with the broken lines. The particle takes a periodic motion around the point $(C_1,S_1)=(1,0)$ on the broken line, so that the electromagnetic fields cannot change the synchrotron and betatron emittances on an average over several revolutions.

On the other hand in the dissipative system, the center point of the periodic motion is located at the point shifted by the vector

$$(\Delta C_1,\Delta S_1)=\left[\frac{\gamma_x-\gamma_\sigma}{2}\ ,\ -\frac{\gamma_x-\gamma_\sigma}{2}\cot\pi q_x\right]$$

from the point $(C_1,S_1)=(1,0)$ [see Eq. (20)]. The center point deviates from the boundary line of the damping region for a more strongly damping oscillation between the betatron and synchrotron oscillations, and vice versa. The case for growing oscillations follows likewise. The shift of the center point leads to the result that the damping rates change to

$$\gamma_x=\gamma_{xd}-\left[\frac{P}{1+P+Q}\right](\gamma_{xd}-\gamma_{\sigma d})\ , \tag{46}$$

$$\gamma_\sigma=\gamma_{\sigma d}+\left[\frac{Q}{1+P+Q}\right](\gamma_{xd}-\gamma_{\sigma d})\ , \tag{47}$$

where

$$P=-\frac{b}{4S_0^3}(C_0-\lambda S_0)\ , \tag{48}$$

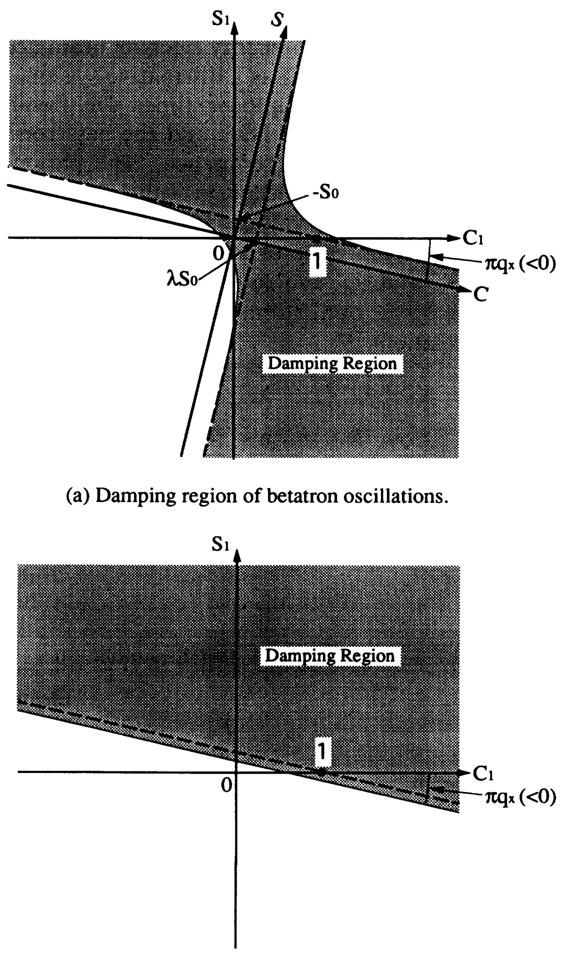
$$Q=\frac{bT}{2\pi\beta_0\nu_s}\left[\frac{\varepsilon_x}{\varepsilon_\sigma}\right]\ , \tag{49}$$

$$C_0=\cos(\pi q_x)\ , \tag{50}$$

and S_0 is defined by Eq. (39). As shown in Eqs. (46) and (47), the system considered is not ruled by Robinson's theorem that $\gamma_x+\gamma_\sigma$ is constant, independent of any electromagnetic field. The quantity C_0 nearly equals unity for the synchro-betatron sum resonance, and usually $|\lambda|\ll 1$, with the result that $C_0-\lambda S_0>0$. The quantity S_0 is negative for synchro-betatron sum resonance, and the quantity b is positive for a positive momentum compaction factor [see Eqs. (35), (19), and (A7)]. Thus the quantities P and Q have positive values. Therefore, Eqs. (46) and (47) indicate that electromagnetic fields have a tendency to equalize the damping or growing rates of synchrotron and betatron oscillations.

As shown in Eqs. (33) and (44), the electromagnetic fields have a large effect on the synchrotron and betatron emittances as the values of t and T become large, i.e., the fractional part of the betatron tune becomes small. This is because the betatron modulation of the synchrotron oscillation depends on the previous betatron motions, so that total effects from the past to the present become larger as the fractional part of the betatron tune becomes smaller.

When the deflecting field is used to excite the synchro-betatron interactions, the deflecting field should be synchronized with the accelerating field so that its frequency is an integer or half-integer multiple of the frequency of the accelerating field. When the deflecting field has an integer times the frequency higher than the accelerating field, the betatron tune should nearly equal an integer resonance. Conversely, when the deflecting field has a half-integer times the frequency higher than the accelerating field, equalization of the damping rates can be realized



(a) Damping region of betatron oscillations.

(b) Damping region of synchrotron oscillations.

FIG. 1. Damping regions of betatron and synchrotron oscillations in the space (C_1,S_1). (a) Damping region of betatron oscillations. (b) Damping region of synchrotron oscillations.

when the betatron tune nearly equals a half-integer resonance and when the accelerating field has an odd harmonic number. Furthermore the synchronous phase of the deflecting field should be set about a zero-crossing phase of the magnetic field in any case.

IV. PARTICLE SIMULATION

The factors S_1 and C_1, which determine the effect of the electromagnetic field on the behavior of particles, cannot be obtained analytically, so that a particle simulation is useful to understand the phenomenon exactly. As an example, particle simulation was applied to the case of the synchro-betatron interactions being induced by deflecting fields. We show how the damping rates of the synchrotron and betatron oscillations depend on the deflecting fields.

A. Algorithm

The algorithm for the particle simulation is briefly summarized below. It can be derived from the canonical equations described with the Hamiltonian except for the dissipative terms [7]. Suppose that an accelerating cavity and a deflecting cavity are installed in a storage ring. The change of each physical quantity is expressed with a Δ sign before the variable as in the following.

(1) At the accelerating cavity,

$$\Delta\phi=\frac{k}{\beta_x\beta_0^5\gamma_0^2}(\zeta x-\eta y)\Delta\delta \ , \tag{51}$$

$$\Delta\delta=v\sin(\phi_{\text{sa}}+\phi+\Delta\theta_x) \ , \tag{52}$$

$$\Delta x=-\eta\Delta\delta_p \ , \tag{53}$$

$$\Delta p_x=-P_\eta\Delta\delta_p \ . \tag{54}$$

(2) At the deflecting cavity,

$$\Delta\phi=\frac{k}{\beta_x\beta_0^5\gamma_0^2}(\zeta x-\eta y)\Delta\delta+\frac{k\eta}{\beta_0^3}\left[1-\frac{\delta}{\beta_0^2\gamma_0^2}\right]\Phi_H \ , \tag{55}$$

$$\Delta\delta=\frac{\partial v}{\partial x}(x+\eta\delta)\sin(\phi_{\text{sd}}+\phi+\Delta\theta_x) \ , \tag{56}$$

$$\Delta x=-\eta\Delta\delta_p \ , \tag{57}$$

$$\Delta p_x=-P_\eta\Delta\delta_p+\Phi_H \ . \tag{58}$$

(3) At the other elements,

$$x_2=M_{11}x_1+M_{12}p_{x1}(1-2\gamma_{xd}) \ , \tag{59}$$

$$p_{x2}=M_{21}x_1+M_{22}p_{x1}(1-2\gamma_{xd}) \ , \tag{60}$$

$$\Delta\phi=\frac{2\pi h\eta_m}{\beta_0^2}\delta_1 \ , \tag{61}$$

$$\delta_2=\delta_1(1-2\gamma_{\sigma d}) \ , \tag{62}$$

where ϕ is the rf phase displacement from the synchronous phase, δ is the relative horizontal deviation, η is the horizontal dispersion , and

$$\zeta=\alpha_x\eta+\beta_xP_\eta \ , \quad P_\eta=\beta_0^2\frac{d\eta}{ds} \ ,$$

$$\Delta\delta_p=\frac{\Delta\delta}{\beta_0^2}\left[1-\frac{\delta}{\beta_0^2\gamma_0^2}\right] \ ,$$

$$\Delta\theta_x=\frac{k}{\beta_0^3}\left[1-\frac{\delta}{\beta_0^2\gamma_0^2}\right](P_\eta x-\eta p_x) \ ,$$

where $v=qV/E_0$ (q is the electric charge of particle, V is the rf voltage, and E_0 is the beam energy), and

$$\Phi_H=\beta_0\frac{qV_H}{E_0}\cos(\phi_{sd}+\phi+\Delta\theta_x) \ , \quad V_H=k\frac{\partial V}{\partial x} \ ,$$

where ϕ_{sa} is the synchronous phase of an accelerating voltage, ϕ_{sa} is the synchronous phase of a deflecting voltage, M_{11}, M_{12}, M_{21}, and M_{22} are components of the second-order transfer matrix for the elements, h is the harmonic number of the accelerating frequency, and the subscripts 1 and 2 in x_1, x_2, p_{x1}, p_{x2}, δ_1, and δ_2 indicate quantities at the entrance and exit positions of the elements.

B. Simulation results and discussions

Here we exemplify a high-energy electron beam accompanied by radiation damping as a dissipative system. The deflecting field has a half-integer times the frequency higher than the accelerating field, and the betatron tune nearly equals a half-integer resonance. The parameters used in the simulation are listed in Table I, and Fig. 2 shows how the deflecting voltage V_H affects the relation between the damping rates of synchrotron and betatron oscillations, γ_x and γ_σ, and the damping partition number of the betatron oscillations, J_x. When $V_H=0$, the well-known relations [9]

$$\gamma_x\propto J_x \ , \quad \gamma_\sigma\propto 3-J_x$$

hold which follow Robinson's theorem that the sum $\gamma_x+\gamma_\sigma$ is constant. However, these damping rates under the deflecting field disobey Robinson's theorem when the betatron tune is on the synchro-betatron sum resonance or in its neighborhood. The simulation results shown in

TABLE I. Parameters for particle simulation.

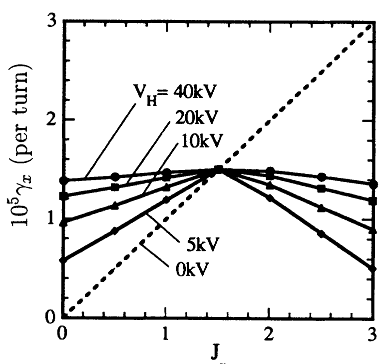
Particle		Electron
Beam energy	E_0	600 MeV
Betatron tune	ν_x	1.496
Synchrotron tune	ν_s	3.85×10^{-3}
Harmonic number	h	7
Momentum compaction factor	α_m	0.2
Twiss parameters (at accelerating cavity)	α_x	−0.8
	β_x	4 m
(at deflecting cavity)	α_x	−0.5
	β_x	3 m
Accelerating voltage	V	40 kV
Accelerating frequency		150 MHz
Deflecting frequency		525 MHz

Fig. 2 support our theory that the deflecting field has a tendency to equalize the damping rates of synchrotron and betatron oscillations. When the damping partition number J_x of the betatron oscillations is less than $\frac{3}{2}$, the radiation damping rate of the betatron oscillations can be increased by a factor of $3/(2J_x)$, which means that the equilibrium emittance of the betatron oscillations can be lowered by a factor of $2J_x/3$.

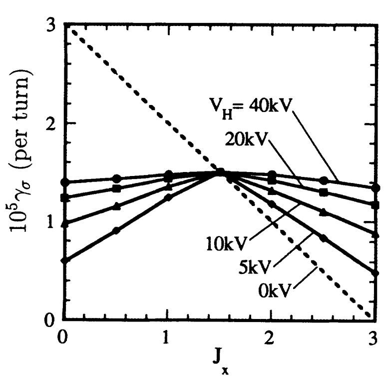
As mentioned above, strong synchro-betatron interactions also can be induced by the horizontal dispersion at the accelerating cavity. We investigated the equalization effect of damping rates of synchrotron and betatron oscillations in the cases of the deflecting field and the horizontal dispersion at the accelerating cavity. We evaluated the equalization effect by the following index:

$$F=\frac{\gamma_x\gamma_\sigma}{\gamma_{xd}\gamma_{\sigma d}} . \tag{63}$$

Figures 3(a) and 3(b) show the betatron tune dependence of the index F under synchro-betatron interactions due to the deflecting field and the horizontal dispersion at the accelerating cavity, respectively, with the unperturbed dissipative damping rates $\gamma_{xd}=0.5\times10^{-5}$ / turn and $\gamma_{\sigma d}=2.5\times10^{-5}$ / turn. The index F has a value of 1.8 for the complete equalization. The betatron tune is set near a half-integer for the deflecting field, although it can be set near an integer as described before, while it should be set near an integer for the horizontal dispersion at the accelerating cavity. Figure 3(a) shows tune dependencies of the index F with the deflecting voltages $V_H=5$, 10, 20, and 40 kV. The sharp spikes at $\nu_x=1.489$ and 1.498 for $V_H=40$ kV correspond to higher-order synchro-betatron resonances $\nu_x+3\nu_s=1.5$ and $2\nu_x+\nu_s=3.0$, respectively. In Fig. 3(b), the solid and broken lines indicate the results with the horizontal dispersion at the accelerating cavity $\eta=1$ and 5 m, respectively. In both cases shown in Figs. 3(a) and 3(b), the index F increases as the betatron tune approaches the integer according to our theory. However, the index F decreases remarkably when the betatron tune continues to approach an integer or half-integer beyond the synchro-betatron resonance. The remarkable decrease of the index F seems to be due to the synchrotron modulation of the betatron oscillations not considered in our theory. The synchrotron modulation of betatron motions is induced either by a sudden change of the closed orbit due to a longitudinal kick in an accelerating field with a finite horizontal dispersion, or by a change of the transverse kick due to an rf phase shift of a

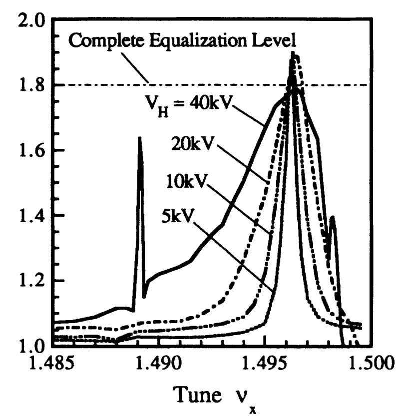


(a) Damping rate of betatron oscillations.

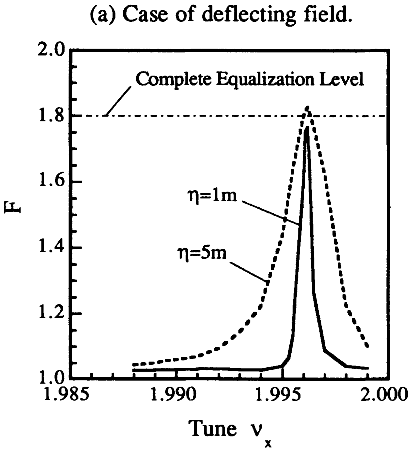


(b) Damping rate of synchrotron oscillations.

FIG. 2. Effect of deflecting voltage V_H on the betatron and synchrotron oscillations. (a) Damping rate of betatron oscillations. (b) Damping rate of synchrotron oscillations.



(a) Case of deflecting field.



(b) Case of horizontal dispersion at the accelerating cavity.

FIG. 3. Tune dependencies of the index F under synchro-betatron interactions due to deflecting field and horizontal dispersion at the accelerating cavity. (a) Case of deflecting field. (b) Case of horizontal dispersion at the accelerating cavity.

deflecting field where the rf phase shift is caused by the synchrotron motion. Independently of the theory presented in this paper, the particles' motions can be almost strictly analyzed using Eqs. (51)–(62) in the numerical simulation, so that the numerical simulation contains the effects of synchrotron modulation of betatron motions.

In the case of deflecting field, curves of 20 and 40 kV vanish when the betatron tune is near the value of 1.5. These indicate beam instabilities due to the synchrotron modulation of betatron motions.

We can see the following points from these results. In the case of the synchro-betatron resonance due to the horizontal dispersion at the accelerating cavity, the betatron tune is restricted within the narrow tuning width of 1×10^{-3} unless the horizontal dispersion has a large value over about 5 m at the accelerating cavity. On the other hand, when using deflecting fields, the betatron tune need not be so strictly set on the synchro-betatron sum resonance; it works even in its neighborhood, so that the tuning width of the betatron tune is as broad as 1×10^{-2}.

We adopted radiation damping as an example of a dissipative system. Equations (59) and (60) show that the algorithm contains only damping rates as information about the dissipative system. Therefore, the algorithm does not depend on the kind of dissipative system, and then the results of the particle simulation do not depend on the damping mechanism, so that they are applicable to any dissipative system.

The synchro-betatron sum resonance induces a push-pull beating wave between synchrotron and betatron oscillations, while keeping the total oscillation energy. Therefore the reciprocal modulations of the synchrotron and betatron oscillations are moderate and not so harmful [4], while the synchro-betatron difference resonance leads to growing instability with the result of beam disruption. Then the strong synchro-betatron interactions described in this paper can be used with no problem.

The equalization of longitudinal and transverse cooling rates can be applied in several cooling methods: (1) synchrotron radiation cooling, (2) stochastic cooling of a bunched beam, and (3) laser cooling of a relativistic ion beam. For (1), a synchrotron radiation brightness can be enhanced by a factor of 1.5 for a large electron storage ring as a low emittance synchrotron radiation source with the damping partition number of the horizontal betatron oscillation J_x of unity. For (2), longitudinal and transverse stochastic cooling can be simultaneously executed by using either a longitudinal or a transverse cooling system. Generally, the transverse beam cooling rate is lower than the longitudinal one, and thus the total cooling time can be reduced when using double-cooling systems for longitudinal and transverse directions. For (3), conventional laser cooling of a relativistic ion beam is possible only in the longitudinal direction, while it is practically impossible in the transverse direction. This is because a large enough transverse interaction region between the ion and laser beams cannot be obtained, and because fluctuations of particle momentum due to random directions of spontaneous emission contribute to stochastic diffusion, leading to transverse heating in a longitudinal cooling system. Equalization of longitudinal and transverse cooling rates by strong synchro-betatron interaction makes transverse and longitudinal laser cooling possible for a relativistic ion beam in a bunching style.

V. CONCLUSIONS

From theoretical investigation and particle simulation, we showed that the damping rates of synchrotron and betatron oscillations can be changed due to a betatron modulation of synchrotron oscillations generated by synchro-betatron interactions so as to have a tendency to equal each other in a dissipative system. The betatron modulation of synchrotron oscillations occurs strongly on the synchro-betatron sum resonance or in its neighborhood by either a longitudinal kick due to an rf electric field in an accelerating cavity with a finite horizontal dispersion, or a transverse kick due to an rf magnetic field in a deflecting cavity. The tuning width of the betatron tune permitting equalization of damping rates is broader for the transverse kick than the longitudinal kick.

The equalization of longitudinal and transverse beam-cooling rates can be applied to several cooling methods. In particular, for radiation cooling, the synchrotron radiation brightness can be enhanced by a factor of 1.5 for a large electron storage ring. This equalizing phenomenon permits simultaneous stochastic cooling in longitudinal and transverse directions in a longitudinal or transverse cooling system. Moreover, longitudinal and transverse laser cooling becomes possible for a relativistic ion beam.

APPENDIX: HAMILTONIAN

Suppose a synchro-betatron interactions are induced by either a longitudinal kick due to an rf electric field in an accelerating cavity with a finite horizontal dispersion, or a transverse kick due to an rf magnetic field in a deflecting cavity.

The system is composed of static magnetic fields and rf electromagnetic fields, including error magnetic fields generating a closed-orbit distortion. Let us consider the case with $\beta_0=1$, for simplicity. The Hamiltonian of the system can be written as

$$H(x,p_x,\sigma,\delta)=H_m+H_c+H_{\rm rf}+H_1+H_2\ , \tag{A1}$$

$$H_m=\frac{p_x^2}{2}+\frac{K}{2}X^2+\frac{\lambda_0}{6}X^3\ , \tag{A2}$$

$$H_c=X\left[\frac{dP_\eta}{ds}\delta+\frac{dP_{Xc}}{ds}-\kappa\delta\right]+\frac{\Delta b}{2\kappa}(1+\kappa X)^2 -\frac{dP_\eta}{ds}\left[\frac{\eta}{2}\delta^2+X_c\delta\right]\ , \tag{A3}$$

$$H_{\rm rf}=-\hat{E}_a(\cos\Psi_a+\Psi_a\sin\phi_{sa})-X\frac{\partial\hat{E}_s}{\partial X}\cos\Psi_d -P_X\hat{E}_x\cos\Psi_d+\frac{\hat{E}_x^2}{2}\cos^2\Psi_d\ , \tag{A4}$$

$$H_1=\frac{(\kappa X-\delta)}{2(1+\delta)}(P_x-\hat{E}_x\cos\Psi_d)^2\ , \tag{A5}$$

$$H_2=\frac{(1+\kappa X)}{8(1+\delta)^3}(P_X-\hat{E}_x\cos\Psi_d)^2\ , \tag{A6}$$

where $X=x+\eta\delta+X_c$ (horizontal displacement), $P_X=p_x+P_\eta\delta+P_{X_c}$, X_c is the transverse displacement of a closed orbit, P_{X_c} is the conjugate momentum of X_c, and

$$K=\kappa^2+\frac{1}{B_0\rho}\left[\frac{\partial B}{\partial X}\right]_0\ ,\quad \lambda_0=\frac{1}{B_0\rho}\left[\frac{\partial^2 B}{\partial X^2}\right]_0\ ,$$

where κ is the curvature of a reference orbit, $B_0\rho$ is the magnetic rigidity, $(\partial B/\partial X)_0$ is the quadrupole magnetic field at the reference orbit, $(\partial^2 B/\partial X^2)_0$ is the sextupole magnetic field at the reference orbit, $\Delta b=\Delta B/B_0\rho$ (ΔB is the error magnetic field in a vertical direction), and

$$\Psi_a=\phi_{sa}+\phi+\Delta\theta_x\ ,\quad \Psi_d=\phi_{sd}+\phi+\Delta\theta_x\ ,$$

$$\hat{E}_a=\frac{eE_a}{kE_0}\ ,\quad \hat{E}_s=\frac{eE_s}{kE_0}\ ,\quad \hat{E}_x=\frac{eE_x}{kE_0}\ ,$$

where E_a is the longitudinal electric field of the accelerating field, E_s is the longitudinal electric field of the deflecting field, E_x is the transverse electric field of the deflecting field, E_0 is the beam energy, and k is the wave number of the electromagnetic field at the point s. Using the above Hamiltonian, we obtain

$$b_0^{\delta\delta}=-\kappa\eta\ , \tag{A7}$$

$$a_1^{\phi}=r\left[\hat{\xi}\hat{E}_a\cos\Psi_{a0}+\left[\frac{\partial\hat{E}_s}{\partial X}-\frac{\alpha_x}{\beta_x}\hat{E}_x\right]\sin\Psi_{d0}\right]\ , \tag{A8}$$

$$b_1^{\phi}=r\left[\frac{\hat{E}_x}{\beta_x}\sin\Psi_{d0}-\hat{\eta}\hat{E}_a\cos\Psi_{a0}\right]\ , \tag{A9}$$

$$\begin{aligned}a_1^{\delta}=&\,r[(1-\kappa\eta)\Delta b+\lambda_0\eta X_c+\kappa P_\eta P_{X_c}]\\&+r\frac{\alpha_x}{\beta_x}[(1-\kappa\eta)P_{X_c}-P_\eta\kappa X_c]\\&+r\left[\hat{\xi}\hat{E}\sin\Psi_{d0}-\frac{\alpha_x}{\beta_x}\hat{E}_x\cos\Psi_{d0}\right]+a_1^{\delta\delta}\delta\ ,\end{aligned} \tag{A10}$$

$$\begin{aligned}b_1^{\delta}=&\,\frac{r}{\beta_x}[\kappa P_\eta X_c-(1-\kappa\eta)P_{X_c}]\\&+r\left[\frac{\hat{E}_x}{\beta_x}\cos\Psi_{a0}-\hat{\eta}\hat{E}\sin\Psi_{a0}\right]+b_1^{\delta\delta}\delta\ ,\end{aligned} \tag{A11}$$

where

$$a_1^{\delta\delta}=r\lambda_0\eta^2+rP_\eta\left[\kappa P_\eta+2\frac{\alpha_x}{\beta_x}(1-\kappa\eta)\right]\ , \tag{A12}$$

$$b_1^{\delta\delta}=-2r\frac{P_\eta}{\beta_x}(1-\kappa\eta)\ , \tag{A13}$$

$$\Psi_{a0}=\phi_{sa}+\phi\ ,\quad \Psi_{d0}=\phi_{sd}+\phi\ , \tag{A14}$$

$$\hat{E}=\eta\frac{\partial\hat{E}_s}{\partial X}+P_\eta\hat{E}_x\ , \tag{A15}$$

$$\hat{\eta}=k\frac{\eta}{\beta_x}\ ,\quad \hat{\xi}=k\frac{\xi}{\beta_x}\ , \tag{A16}$$

and r is the betatron amplitude.

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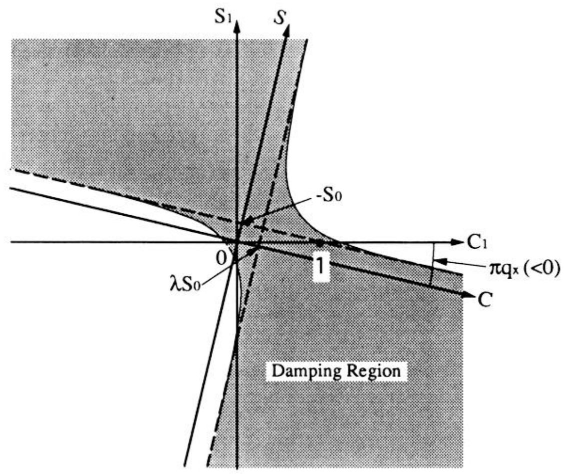
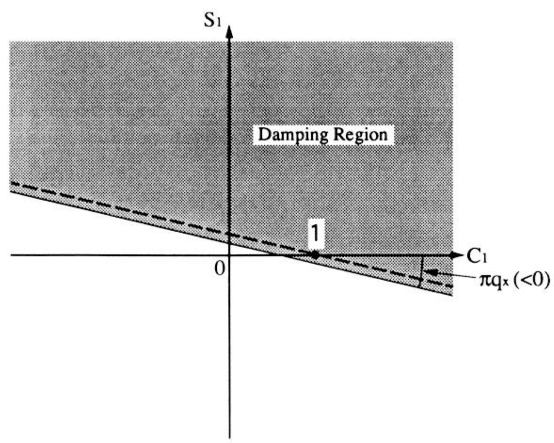


(a) Damping region of betatron oscillations.



(b) Damping region of synchrotron oscillations.

FIG. 1. Damping regions of betatron and synchrotron oscillations in the space (C_1, S_1). (a) Damping region of betatron oscillations. (b) Damping region of synchrotron oscillations.